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LETTER TO THE EDITOR

# A disorder solution for a generalised mixed-spin model

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**Abstract.** A generalised mixed-spin model on a checkerboard square lattice is solved exactly along a trajectory in the parameter space by the use of an exact decimation method. A closed-form expression of the free energy per site is obtained and a criterion determining the validity of the solution is established.

In recent years, much interest has been devoted to the study of disorder solutions for a number of models, e.g., two-dimensional anisotropic Ising and Potts models with a field (Rujan 1984, Baxter 1984, Jaekel and Maillard 1985b, Wu 1985), three-dimensional Ising model (Welberry and Miller 1978, Jaekel and Maillard 1985a) and the general eight-vertex model (Giacomini 1986). So far, three methods, the statistical theory of Markov processes, the transfer-matrix technique of statistical mechanics and the exact decimation procedure, have been used for obtaining disorder solutions. The solutions provide very useful information, including constraints on the analytical behaviour of the partition function, an infinite number of $n$-point correlation functions and many other quantities such as susceptibility at the disorder variety. Moreover, Georges *et al* (1986) have recently shown that it is also possible to obtain information in the vicinity of disorder solutions through a new type of perturbative expansion.

In this letter, the exact decimation method, as used by Jaekel and Maillard and by Wu, is applied to the generalised mixed-spin (GMS) model (Tang and Hu 1986) which, composed of two interpenetrating sublattices occupied by spin $\frac{1}{2}$ and spin 1 respectively, contains the pure Ising model, the mixed-spin (MS) model (Schofield and Bowers 1980) and a particular site-diluted Ising model (vacancy only occupies one sublattice) as special cases. In spite of its lower translational symmetry, it has been shown that the GMS model exhibits an Ising transition (Tang and Hu 1986). The GMS model can be solved exactly on the honeycomb lattice, by transforming it into a triangular Ising model with nearest-neighbour interactions via star-triangle relation, but cannot on the square lattice. Therefore, it is not without interest to seek the disorder solution for the square-lattice GMS model.

Consider the GMS model on a general checkerboard square lattice, shown in figure 1, of $2N$ sites with the Hamiltonian

$$\mathscr{H}\{\sigma, S\} = \sum_{\blacksquare} (-J_1\sigma_1 S_2 - J_2\sigma_3 S_2 - J_3\sigma_3 S_4 - J_4\sigma_1 S_4) - G\sum_i S_i^2 \qquad (1)$$

where $\sigma = \pm\frac{1}{2}$ and $S = 0, \pm 1$, the summation $\Sigma_{\blacksquare}$ is taken over all shaded square faces of the lattice and each shaded square is bordered by interactions $-J_1$, $-J_2$, $-J_3$ and

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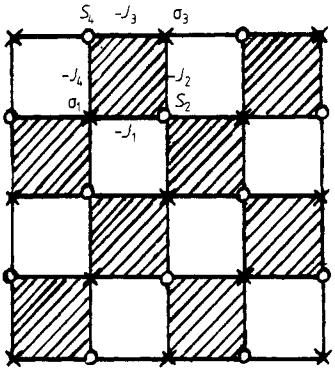


**Figure 1.** General checkerboard lattice with periodic condition in the horizontal direction. The GMS model consists of spin-1 and spin-$\frac{1}{2}$ Ising objects, which are indicated by open circles (○) and crosses (×) respectively. Each shaded square is bordered by interactions $-J_1$, $-J_2$, $-J_3$, and $-J_4$.

$-J_4$ (cf figure 1). The free energy per site is

$$f = \ln \lim_{2N\to\infty} (Z)^{1/2N} \tag{2}$$

where $Z$ is the partition function defined by (1).

To perform the decimation procedure, a periodic boundary condition is imposed in the horizontal direction and the Hamiltonian (1) is modified by changing the parameters $G$ and $J_1$ associated with the spins located on the upper boundary to new values $G'$ and 0 respectively. The modified Hamiltonian will have the same bulk free energy as the original one provided that $G'$ and $J_1$ are real, which is a sufficient condition as pointed out by Chao and Wu (1985). We carry out the spin sums over the first row of spins in figure 1 with the modified Hamiltonian and require

$$\exp(\Delta' S_4^2 + K_1\sigma'_1 S_2 + K_2\sigma'_3 S_2 + K_3\sigma'_3 S_4 + K_4\sigma'_1 S_4) = F\exp(\Delta^* S_2^2) \tag{3}$$

where $K_i = \frac{1}{2}J_i/kT$, $\Delta' = G'/kT$, $\Delta^* = G^*/kT$ and $\sigma'_i = 2\sigma_i = \pm 1$. The remaining lattice is an exact copy of the original one except that it has one less row and the parameter $G$ associated with the new boundary spins takes the value $G + G^*$. If we require further

$$G' = G + G^* \tag{4}$$

then the decimation process can be performed continually by summing the new boundary spins. Finally, all the spins are decimated except those in the last row, which, giving rise to a positive factor, can be neglected in the bulk limit. Since each decimated square contributes a factor $F$ to the partition function, we finally obtain from (2) and (3) the solution

$$\begin{aligned} f &= (F)^{1/2} \\ &= \{2[2\exp(\Delta')\cosh K_3 \cosh K_4] + 1\}^{1/2} \end{aligned} \tag{5}$$

and

$$\exp(\Delta') = \frac{\sinh K_1 \cosh K_2}{\cosh(K_4 - K_2)\sinh(K_3 - K_1) - \cosh(K_4 - K_2)\sinh(K_3 - K_1)} \tag{6}$$

which is valid along the trajectory

$$\exp(\Delta)=\exp(\Delta')[2\exp(\Delta')\cosh K_3\cosh K_4+1]/\exp(K_1)\{\exp(\Delta')[\exp(K_3)\times\cosh(K_4+K_2)+\exp(-K_3)\cosh(K_2-K_4)]+\cosh K_2\} \quad (7)$$

and confined to the regions

$$J_1J_2J_3J_4<0 \quad (8)$$

$$|J_1|<|J_2|,|J_3|,|J_4| \quad (9)$$

and

$$T\leqslant T_D \quad (10)$$

where $T_D$ is the temperature defined by

$$\frac{\cosh(|K_4|+|K_2|)}{\cosh(|K_4|-|K_2|)}=\frac{\sinh(|K_3|+|K_1|)}{\sinh(|K_3|-|K_1|)}. \quad (11)$$

The conditions (8)-(10) derived from the requirement that $G'$ is real indicate that the decimations must be used with care and that it is valid only when carried out along one preferred lattice direction as in the case of a triangular Ising model in a non-zero magnetic field considered by Wu (1985).

It is interesting to see the case of $T=T_D$. It follows from (6), (8), (9) and (11) that $G$ becomes infinite when $T=T_D$, and hence the GMS model reduces to a checkerboard Ising model, which is the case of $q=2$ of the corresponding Potts model considered by Jaekel and Maillard (1984), and Baxter (1984). After eliminating an infinite factor $\exp(\Delta)$, we have

$$f=\{2[\cosh(K_2+K_4)\cosh(K_1+K_3)+\cosh(K_2-K_4)\cosh(K_1-K_3)]\}^{1/2} \quad (12)$$

which can be easily shown to be equivalent to Jaekel and Maillard's equation (5) and Baxter's equation (32) with $q=2$.

In summary, a disorder solution has been obtained for the GMS model on a checkerboard lattice and the constraints imposed by validity conditions of the decimation method on the interactions are discussed. The solution, given by (5) and (6), is valid under the conditions (8)-(10) and reduces to that of the Ising model when $T=T_D$.

## References


Baxter R J 1984 *J. Stat. Phys.* **34** 615
Chao N C and Wu F Y 1985 *J. Phys. A: Math. Gen.* **18** L603
Georges A, Hansel D, Le Doussal L P and Maillard J M 1986 *J. Phys. A: Math. Gen.* **19** 1001
Giacomini H J 1986 *J. Phys. A: Math. Gen.* **19** L335
Jaekel M T and Maillard J M 1984 *J. Phys. A: Math. Gen.* **17** 2079
—— 1985a *J. Phys. A: Math. Gen.* **18** 641
—— 1985b *J. Phys. A: Math. Gen.* **18** 2271
Rujan P 1984 *J. Stat. Phys.* **34** 615
Schofield S L and Bowers R G 1980 *J. Phys. A: Math. Gen.* **23** 3697
Tang K-F and Hu J-Z *Acta Phys. Sin.* **35** 1048
Welberry T R and Miller G H 1978 *Acta Crystallogr.* A **34** 120
Wu F Y 1985 *J. Stat. Phys.* **40** 613